# XI. *On the Double Tangents of a Plane Curve.* By A. Cayley, *Esq.*, *F.R.S.*

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It was first shown by Plücker on geometrical principles, that the number of the double tangents of a plane curve of the order $m$ was $\frac{1}{2}m(m-2)(m^2-9)$: see the note, "Solution d'une question fondamentale concernant la théorie générale des Courbes," *Crelle*, t. xii. pp. 105–108 (1834), and the "Théorie der algebraischen Curven" (1839). The memoir by Hesse, "Ueber die Wendepuncte der Curven dritter Ordnung," *Crelle*, t. xxviii. pp. 97–107 (1844), contains the analytical solution of the allied easier problem of the determination of the points of inflexion of a plane curve. In the memoir, "Recherches sur l'élimination et sur la théorie des Courbes," *Crelle*, t. xxxiv. pp. 30–45 (1847), I showed how the problem of double tangents admitted of an analytical solution, viz. if $U=0$ is the equation of the curve, L, M, N the first derived functions of U, and

$$D=\alpha(M\partial_x-N\partial_y)+\beta(N\partial_y-L\partial_z)+\gamma(L\partial_z-M\partial_x)$$

(where $\alpha$, $\beta$, $\gamma$ are arbitrary), then the points of contact of the double tangents are given as the intersections of the curve $U=0$, with a curve the equation whereof is in the first instance obtained under the form $[Y]=0$; $[Y]$ being a given function of $D^2U$, $D^3U$, ..$D^mU$ of the degree $m^2-m-6$ in respect of $(\alpha, \beta, \gamma)$, the degree

$$m^3-2m^2-10m+12$$

in respect of $(x, y, z)$, and the degree $m^2+m-12$ in respect of the coefficients of U. It was necessary, in order that the points of intersection should be independent of the arbitrary quantities $(\alpha, \beta, \gamma)$ that we should have identically

$$[Y]=\Lambda.U+N.\Pi U,$$

N being of the degree $m^2-m-6$ in $(\alpha, \beta, \gamma)$, and consequently $\Pi U$ a function of $(x, y, z)$ without $(\alpha, \beta, \gamma)$. Guided by Hesse's investigation for the points of inflexion, I asserted that it was probable that N was of the form $(\alpha x+\beta y+\gamma z)^{m^2-m-6}$; which being so, $\Pi U$ would be of the degree $(m-2)(m^2-9)$ in respect of $(x, y, z)$, and the degree $m^2+m-12$ in respect of the coefficients, and I was thus led to the theorem, "On trouve les points de contact des tangentes doubles en combinant avec l'équation de la courbe une équation $\Pi U=0$, de l'ordre $(m-2)(m^2-9)$ par rapport aux variables et de l'ordre $m^2+m-12$ par rapport aux coefficients—c'est à dire, puisqu'il corresponde deux points de contact à une tangente double, le nombre de ces tangentes est égal à $\frac{1}{2}m(m-2)(m^2-9)$: théorème démontré indirectement par M. Plücker."

Hesse, in the memoir "Ueber Curven dritter Ordnung, &c.," *Crelle*, t. xxxvi. pp. 143–176 (1848), showed how the components $D^2U$, $D^3U$, .. $D^mU$ of $[Y]$ could each of them be expressed in a simplified form, and he thus effected the actual reduction of [Y] to the form $\Lambda.\mathrm{U}+(\alpha x+\beta y+\gamma z)^{4(m-3)}\mathrm{R}$, where R still contained the arbitrary quantities $(\alpha, \beta, \gamma)$ in the degree $(m-2)(m-3)$. In particular for a quartic curve, the equation $\mathrm{R}=0$ was shown to be

$$3\mathrm{Q}_2\mathrm{Q}_4-\mathrm{Q}_3^2=0,$$

where the left-hand side is of the degree 2 in $(\alpha, \beta, \gamma)$ and the degree 16 in $(x, y, z)$; and which should therefore by means of the equation $\mathrm{U}=0$ be reducible so as to contain the factor $(\alpha x+\beta y+\gamma z)^2$.

JACOBI's paper, "Beweiss des Satzes, dass eine Curve $n$-ten Grades im allgemeinen $\frac{1}{2}n(n-2)(n^2-9)$ Doppeltangenten hat," *Crelle*, t. xl. pp. 237–260 (1850), did not, I think, materially advance the solution of the question. In a letter to JACOBI, dated the 30th December, 1849, published at the conclusion of the last-mentioned paper, HESSE gave the equation of the curve of the 14th order for the points of contact of the double tangents of a quartic, viz. in my notation,

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \partial_x \mathrm{H}, \partial_y \mathrm{H}, \partial_z \mathrm{H})^2-\mathrm{H}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \partial_x, \partial_y, \partial_z)^2\mathrm{H}=0,$$

and the demonstration is given in HESSE's paper, "Ueber die ganzen homogenen Functionen von der dritten und vierten Ordnung zwischen drei Variabeln," *Crelle*, t. xli. pp. 285–292 (1851), and is reproduced in Mr. SALMON's Treatise on the Higher Plane Curves (1852). Two very interesting memoirs by HESSE and STEINER, *Crelle*, t. xlix. (1855), relate to the *geometrical* theory of the double tangents of a quartic, and it is not necessary to refer to them more particularly. It is to be observed that the curve which determines the points of contact of the double tangents is not absolutely determinate; for we may, it is clear, in the place of $\Pi\mathrm{U}=0$, write $\Pi\mathrm{U}+\mathrm{M}.\mathrm{U}=0$, where M is an arbitrary function of the proper degree: a very elegant transformation in the case of the quartic is given in HESSE's paper, "Transformation der Gleichung der Curven 14ten Grades, welche eine gegebene Curve 4ten Grades in den Berührungspuncten ihrer Doppeltangenten schneiden," *Crelle*, t. lii. pp. 97–103 (1856).

Mr. SALMON's work above referred to, contains the fundamental theorem of the *tangential* of a cubic, viz. a tangent to a cubic meets the cubic in a third point which lies on the second or line polar of the point of contact with respect to the Hessian. In my "Memoir on Curves of the Third Order*," I gave an identical equation relating to the tangential of a cubic, but which is not there exhibited in its proper form; this was afterwards effected by Mr. SALMON, in the paper "On Curves of the Third Order†." The equation, as given by Mr. SALMON, is in the notation of the present memoir,

$$-\mathfrak{H}.\mathrm{U}+\tfrac{1}{3}\mathfrak{D}\mathfrak{H}.\mathrm{DU}-\tfrac{1}{3}\mathrm{DH}.\mathfrak{D}\Upsilon+\mathrm{H}.\Upsilon=0,$$

an equation which in fact puts in evidence the last-mentioned theorem for the tangential of a cubic.

* Philosophical Transactions, vol. cxlvii. (1857), pp. 415–446, art. No. 37.

† Ibid. vol. cxlviii. (1858), pp. 535–541.

The idea occurred to me of considering, in the case of the higher plane curves, the *tangentials* of a given point of the curve, viz. the points in which the tangent again meets the curve; for by expressing that two of these tangentials were coincident, we should have the condition that the given point is the point of contact of a double tangent. But I was not able to complete the solution.

Finally, Mr. SALMON discovered the equation of a curve of the order $m-2$, which by its intersections with the tangent at the given point determines the tangentials, and by expressing that the curve in question is touched by the tangent, he was led to a complete solution of the Double-tangent problem. Mr. SALMON'S result is given in the note, "On the Double Tangents to Plane Curves," in the Philosophical Magazine for October 1858. The discovery just referred to led me to the investigations of the present memoir, in which it will be seen that I obtain, for a curve of any order whatever, the identical equation corresponding to the before-mentioned equation obtained by Mr. SALMON in the case of a cubic; which identical equation puts in evidence the theorem as to the tangentials of the curve, and may thus be considered as containing in itself the solution of the Double-tangent problem: the identical equation is besides interesting for its own sake, as a part of the theory of ternary quantics.

1. Mr. SALMON'S solution of the problem of double tangents is based upon the following analytical determination of the tangentials of any point of the curve.

Let

$$\Upsilon=(*\between \mathrm{X}, \mathrm{Y}, \mathrm{Z})^n=0$$

be the equation of the given curve, (X, Y, Z) being current coordinates; and let $(x, y, z)$ be the coordinates of a point on the curve, so that we have

$$\mathrm{U}=(*\between x, y, z)^n=0,$$

a condition satisfied by the coordinates of the point in question.

Then the tangent

$$\mathrm{V}=(\mathrm{X}\partial_x+\mathrm{Y}\partial_y+\mathrm{Z}\partial_z)\mathrm{U}=0$$

at the point $(x, y, z)$, meets the curve besides in $(n-2)$ points, which are the tangentials of the given point $(x, y, z)$, and which are determined as the intersections of the tangent $\mathrm{V}=0$ with a certain curve,

$$\Omega=(\dagger\between x, y, z)^{n-2}=0.$$

2. To express the equation of this curve, let $\mathrm{U}_1$, $\mathrm{U}_2$, .. be the successive emanants of U, taken with the facients of emanation $(x_{\prime}, y_{\prime}, z_{\prime})$, viz.

$$\mathrm{U}_1=\frac{1}{n}(x_{\prime}\partial_x+y_{\prime}\partial_y+z_{\prime}\partial_z)\mathrm{U},$$

$$\mathrm{U}_2=\frac{1}{n(n-1)}(x_{\prime}\partial_x+y_{\prime}\partial_y+z_{\prime}\partial_z)^2\mathrm{U},$$

.

.

where it should be noticed that the numerical determination is such, that putting $(x, y, z)$ for $(x_{\prime}, y_{\prime}, z_{\prime})$, then $\mathrm{U}_1$, $\mathrm{U}_2$, .. become respectively equal to U. Suppose also that H, $H_1$, $H_2$, ... are the Hessians of U, $U_1$, $U_2$, ..., viz. H is the determinant formed with the second derived functions of U with respect to $(x, y, z)$, $H_1$ the like determinant with the second derived functions of $U_1$ with respect to the same quantities $(x, y, z)$; and so on. Moreover let $D^{n-2}H, =(X\partial_x+Y\partial_y+Z\partial_z)^{n-2}H$, denote the $(n-2)$thic emanant of H with respect to the current coordinates (X, Y, Z) as facients of emanation; and similarly let $D^{n-2}H_1$, $D^{n-2}H_2$, ... denote the $(n-2)$thic emanants of $H_1$, $H_2$, ... in respect to the same facients of emanation—it being understood that in all these functions, $(x_{\prime}, y_{\prime}, z_{\prime})$ are after the differentiations to be replaced by $(x, y, z)$. It is to be observed that $U_r$ is of the degree $(n-r)$ in $(x, y, z)$, and consequently $H_r$ of the degree $3(n-2-r)$; hence $D^{n-2}H_r$ is of the degree $3(n-2-r)-(n-2), =2(n-2)-3r$, which implies that $r \not> \frac{2}{3}(n-2)$, for otherwise $D^{n-2}H_r$ would be identically equal to zero. Upon replacing $(x_{\prime}, y_{\prime}, z_{\prime})$ by $(x, y, z)$, $D^{n-2}H_r$ ($r$ satisfying the above condition) becomes of the degree $2(n-2)$ in $(x, y, z)$, and it is obviously of the degree 3 in the coefficients of U, and of the degree $(n-2)$ in the current coordinates (X, Y, Z).

3. This being premised, we have

$$\Omega=(\dagger \between X, Y, Z)^{n-2}$$

$$=D^{n-2}H-\frac{n-1}{1}D^{n-2}H_1+\&c.=0$$

for the equation of the curve of the order $(n-2)$, which by its intersection with the tangent gives the tangentials of the given point; the numerical coefficients are the binomial coefficients of the order $(n-1)$ taken with the signs + and − alternately, and the series is continued as long as the terms do not vanish, that is, if as before $r$ denote the suffix of H, for so long as $r \not> \frac{2}{3}(n-2)$; but of course the value will not be altered by continuing the series to $r=n-1$. In particular, for the quartic we have

$$\Omega=D^2H-3D^2H_1,$$

for the quintic

$$\Omega=D^3H-4D^3H_1+6D^3H_2,$$

and so on. The function $\Omega$, like the several component terms, is of course of the degree 3 in the coefficients of U, and of the degree $2(n-2)$ in $(x, y, z)$.

4. It is to be remarked that the formula applies to a cubic; we have here simply $\Omega=DH$, which agrees with a result already mentioned. It may be noticed also that in the general case the formula gives at once the condition for the points of inflexion; in fact, if the point $(x, y, z)$ be a point of inflexion, then one of the tangentials must coincide with this point, or the equation $\Omega=0$ will be satisfied by writing therein $(x, y, z)$ for (X, Y, Z); but when this is done $D^{n-2}H$, $D^{n-2}H_1$ &c. reduce themselves (to numerical factors *près*) to H, and the equation becomes simply $H=0$, which is the well-known condition for the points of inflexion.

5. If two of the tangentials coincide, or what is the same thing, if the tangent $V=0$ touches the curve $\Omega=0$, then the point $(x, y, z)$ will be the point of contact of a double tangent. The equation which expresses the condition in question, treating therein

$(x, y, z)$ as current coordinates, is consequently that of a curve, intersecting the given curve (now represented by $U=0$) in the points of contact of the double tangents. The process leads to a determinate form $\Pi U=0$, of the curve in question, but of course any curve whatever, $\Pi U+M.U=0$, will intersect the curve $U=0$ in the points of contact of the double tangents.

6. I write for the moment

$$\begin{aligned}\Omega&=(A,\ldots \between X, Y, Z)^{n-2}=0,\\ V&=\xi X+\eta Y+\zeta Z \qquad =0,\end{aligned}$$

for the two equations; the coefficients $(A,\ldots)$, as already mentioned, are of the degree $2(n-2)$ in $(x, y, z)$ and of the degree 3 in the coefficients of U; or as we may express it,

$$A,\ldots=(a,\ldots)^1(x, y, z)^{2(n-2)}.$$

In like manner $\xi, \eta, \zeta$ are of the degree $(n-1)$ in $(x, y, z)$, and the degree 1 in the coefficients of U, or we may write

$$\xi, \eta, \zeta=(a,\ldots)^1(x, y, z)^{n-1}.$$

7. The equation which expresses that the line $V=0$ touches the curve $\Omega=0$, is $F\Omega=0$, where the facients of the Reciprocant $F\Omega$ are the coefficients $(\xi, \eta, \zeta)$ of the linear function. This equation is of the form

$$(A,\ldots)^{2(n-3)}(\xi, \eta, \zeta)^{(n-2)(n-3)}=0\ ;$$

or attending to the forms of $(A,\ldots)$ and $(\xi, \eta, \zeta)$, it is of the form

$$(a,\ldots)^{6(n-3)+(n-2)(n-3)}(x, y, z)^{4(n-2)(n-3)+(n-1)(n-2)(n-3)}=0,$$

or what is the same thing, the form

$$(a,\ldots)^{(n+4)(n-3)}(x, y, z)^{(n-2)(n^2-9)}=0,$$

viz. the curve through the points of contact of the double tangents is a curve of the order $(n-2)(n^2-9)$, and its equation contains the coefficients of the equation $U=0$ of the given curve in the degree $(n+4)(n-3)$. And since each double tangent corresponds to two points of contact, the number of double tangents is $\frac{1}{2}n(n-2)(n^2-9)$. This agrees with the before-mentioned results.

8. The whole problem is thus reduced to the demonstration of Mr. SALMON's expression for the curve $\Omega=0$. To fix the ideas, consider the case of a quartic curve $\Upsilon=(*\between X, Y, Z)^4=0$, and let the function $U=(*\between x, y, z)^4$ (or as for shortness we may write it, $U=(x, y, z)^4$) and certain of its emanants be represented as follows, viz.—

$$\begin{aligned}a&=(x, y, z)^4 \qquad ,\\ b&=(x, y, z)^3(X, Y, Z)\,,\\ c&=(x, y, z)^2(X, Y, Z)^2,\\ d&=(x, y, z)\,(X, Y, Z)^3,\\ e&=\quad .\quad .\quad (X, Y, Z)^4,\end{aligned}$$

$$\begin{aligned}
a' &= (x,y,z)^3 \quad . \quad . \quad (X',Y',Z')\,,\\
b' &= (x,y,z)^2(X,Y,Z)\,(X',Y',Z')\,,\\
c' &= (x,y,z)\,(X,Y,Z)^2(X',Y',Z')\,,\\
d' &= \quad . \quad . \quad (X,Y,Z)^3(X',Y',Z')\,,
\end{aligned}$$

$$\begin{aligned}
a'' &= (x,y,z)^2 \quad . \quad . \quad (X',Y',Z')^2,\\
b'' &= (x,y,z)\,(X,Y,Z)\,(X',Y',Z')^2,\\
c'' &= \quad . \quad . \quad (X,Y,Z)^2(X',Y',Z')^2,
\end{aligned}$$

where (X′, Y′, Z′) are new arbitrary facients; but, as before, (X, Y, Z) are taken to be current coordinates, and $(x, y, z)$ the coordinates of the given point on the curve:

$e=0$ is the equation of the curve;

$d=0$, the equation of the first or cubic polar of the point $(x, y, z)$;

$b=0$, the equation of the last or line polar of the point $(x, y, z)$, or what is the same thing (the point being on the curve), the tangent of the curve at this point;

$a=0$, the condition which expresses that the point is on the curve.

9. Imagine now an identical equation,

$$\text{I}a+\text{II}b+\text{III}d+\text{IV}e=0\,;$$

then, since $a=0$, we have

$$\text{II}b+\text{III}d+\text{IV}e=0\,;$$

and if in this equation we write $b=0$, $e=0$, it becomes $\text{III}d=0$, that is, the points of intersection of the curve $e=0$ and the tangent $b=0$ lie on one or other of the curves $d=0$, $\text{III}=0$. But the points in question do not lie on the curve $d=0$, consequently they lie on the curve $\text{III}=0$.

10. To explain the law of formation of the multipliers I, II, III, IV, I form the matrix

$$\begin{pmatrix} a, & b, & c, & d\,; & a', & b', & c' \\ b, & c, & d, & e\,; & b', & c', & d' \\ a', & b', & c', & d'\,; & a'', & b'', & c'' \end{pmatrix},$$

and then we have

$$\text{I}=-\begin{vmatrix} d, & c, & b' \\ e, & d, & c' \\ d', & c', & b'' \end{vmatrix}+\begin{vmatrix} d, & b, & c' \\ e, & c, & d' \\ d', & b', & d'' \end{vmatrix},$$

$$\text{II}=-\begin{vmatrix} d, & c, & a' \\ e, & d, & b' \\ d', & c', & a'' \end{vmatrix}-\begin{vmatrix} d, & b, & b' \\ e, & c, & c' \\ d', & b', & b'' \end{vmatrix}-\begin{vmatrix} d, & a, & c' \\ e, & b, & d' \\ d', & a', & c'' \end{vmatrix},$$

$$\text{III}=-\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix}-\begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix}-\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix},$$

$$\mathrm{IV}= \begin{vmatrix} a, & b, & b' \\ b, & c, & c' \\ a', & b', & b'' \end{vmatrix} + \begin{vmatrix} a, & c, & a' \\ b, & d, & b' \\ a', & c', & a'' \end{vmatrix},$$

values which, as I proceed to show, satisfy the identical equation

$$\mathrm{I}a+\mathrm{II}b+\mathrm{III}d+\mathrm{IV}e=0.$$

11. We have in fact

$$\begin{aligned} \mathrm{I}= \ & d\,(db''-c'^2+cc''-b'd') \\ &+e\,(b'c'-b''c+b'c'-bc'') \\ &+d'(cc'-db'+bd'-cc'), \end{aligned}$$

where the last line is

$$bd'^2-db'd';$$

$$\begin{aligned} \mathrm{II}= \ & d(b'c'-a''d+b'c'-b''c+a'd'-bc'') \\ &+e\,(ca''-a'c'+bb''-b'^2+ac''-a'c') \\ &+d'(a'd-b'c+b'c-bc'+bc'-ad'), \end{aligned}$$

where the last line is

$$da'd'-ad'^2;$$

$$\begin{aligned} \mathrm{III}= \ & a\,(-b'd'+cc''+c'^2-b''d+b'd'-ea'') \\ &+b\,(bc''-b'c'+b''c-b'c'+a''d-a'd') \\ &+a'(cc'-bd'+b'd-cc'+a'e-b'd), \end{aligned}$$

where the last line is

$$-ba'd'+ea'^2;$$

and

$$\begin{aligned} \mathrm{IV}= \ & a(b''c-b'c'+da''-b'c') \\ &+b\,(b'^2-bb''+a'c'-a''c) \\ &+a'(bc'-b'c+cb'-a'd), \end{aligned}$$

where the last line is

$$ba'c'-d'^2.$$

And these values may be expressed as follows:—

$$\begin{aligned} \mathrm{I}= \ & a(\ 0\ ) & \\ &+b(\ d'^2\ ) & (12) \\ &+d(\ b''d+c''c-b'd'-c'c'-d'b') & (13) \\ &+e(-b''c-c''b+b'c'+c'b'\ ), & (14) \end{aligned}$$

$$\begin{aligned} \mathrm{II}= \ & a(-d'^2\ ) & (21) \\ &+b(0\ ) & \\ &+d(-a''d-b''c-c''b+a'd'+b'c'+c'b'+d'a') & (23) \\ &+e(\ a''c+b''b+c''a-a'c'-b'b'-c'a'\ ), & (24) \end{aligned}$$

$$
\begin{array}{llr}
\text{III}= & a(-b''d-c''d+b'd'+c'c'+d'b' \qquad\qquad ) & (31)\\
 & +b(\ a''d+b''c+c''b-a'd'-b'c'-c'b'-d'a') & (32)\\
 & +d(\ 0 \qquad\qquad\qquad\qquad\qquad\qquad ) & \cdot\\
 & +e(-a''a+a'a' \qquad\qquad\qquad\qquad ), & (34)\\
\text{IV}= & a(\ b''c+c''b-b'c'-c'b' \qquad\qquad ) & (41)\\
 & +b(-a''c-b''b-c''a+a'c'+b'b'+c'a' \quad ) & (42)\\
 & +d(\ a''a-a'a' \qquad\qquad\qquad\qquad ) & (43)\\
 & +e(\ 0 \qquad\qquad\qquad\qquad\qquad\qquad ), & \cdot
\end{array}
$$

which are of the form

$$
\begin{aligned}
\text{I}&=a\ 0\ +b(12)+d(13)+e(14),\\
\text{II}&=a(21)+b\ 0\ +d(23)+e(24),\\
\text{III}&=a(31)+b(32)+d\ 0\ +e(34),\\
\text{IV}&=a(41)+b(42)+d(43)+e\ 0\ ,
\end{aligned}
$$

where $(12)=-(21)$ &c., and which therefore satisfy the equation

$$\text{I}a+\text{II}b+\text{III}d+\text{IV}e=0.$$

12. The equation of the curve which by its intersection with the tangent gives the tangentials, is

$$
\text{III}=-\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix}-\begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix}-\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix}=0,
$$

the degrees of which are

in the coefficients of U, 3,
in $(x\ , y\ , z\ )$ . . . 6,
in (X, Y, Z ) . . . 4,
in (X′, Y′, Z′) . . . 2;

and it only remains to divest this equation of a factor which it contains,

$$
\begin{vmatrix} x\ , & y\ , & z \\ \text{X}, & \text{Y}, & \text{Z} \\ \text{X}', & \text{Y}', & \text{Z}' \end{vmatrix}^2,
$$

which being thrown out, the equation will be independent of (X′, Y′, Z′) and of the degrees

in the coefficients of U, 3,
in $(x\ , y\ , z\ )$ . . . . 4,
in (X, Y, Z) . . . . 2,

and will in fact be the before-mentioned equation $\Omega=\text{D}^2\text{H}-3\text{D}^2\text{H}_1=0$.

13. Write for shortness,

$$\begin{vmatrix} x\,, & y\,, & z \\ \mathrm{X}, & \mathrm{Y}, & \mathrm{Z} \\ \mathrm{X}', & \mathrm{Y}', & \mathrm{Z}' \end{vmatrix} = \Lambda,$$

it is to be shown that

$$\mathrm{III} = -\tfrac{1}{2}\Lambda^2(\mathrm{D}^2\mathrm{H} - 3\mathrm{D}^2\mathrm{H}_1).$$

14. To effect this I remark that we have identically

$$\begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix} = \Lambda^2\mathrm{H};$$

and I proceed to operate upon this equation with $\mathrm{D} = \mathrm{X}\partial_x + \mathrm{Y}\partial_y + \mathrm{Z}\partial_z$.

I notice that

$$a, b, c, d, e;\quad a', b', c', d';\quad a'', b'', c''$$

are in regard to $(x, y, z)$ of the degrees

$$4,\quad 3, 2, 1, 0;\qquad 3,\quad 2, 1, 0;\qquad 2,\quad 1, 0;$$

or what is the same thing, since for the case in hand $n=4$, of the degrees

$$n, n-1, \ldots\ ;\quad n-1, n-2, \ldots;\quad n-2, n-3, .$$

and we have

$$\mathrm{D}a = nb,\ \mathrm{D}b = (n-1)c, \ldots\quad \mathrm{D}a' = (n-1)b',\ \mathrm{D}b' = (n-2)c', \ldots\quad \mathrm{D}a'' = (n-2)b'', \ldots$$

15. In the determinant

$$\begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix}, = \Lambda^2\mathrm{H},$$

the degrees of the terms (other than each top term, the degree of which is higher by unity) in the several columns are $n-1$, $n-2$, $n-2$; if then we operate on the determinant with D, and as regards the top terms we write

$$\begin{aligned} \mathrm{D}a &= b + (n-1)b, \\ \mathrm{D}b &= c + (n-2)c, \\ \mathrm{D}a' &= b' + (n-2)b', \end{aligned}$$

we have in the first place a term

$$\begin{vmatrix} b\,, & c\,, & b' \\ b\,, & c\,, & b' \\ a', & b', & a'' \end{vmatrix}$$

which vanishes, and next the terms

$$(n-1)\begin{vmatrix} b\,, & b\,, & a' \\ c\,, & c\,, & b' \\ b', & b', & a'' \end{vmatrix} + (n-2)\begin{vmatrix} a\,, & c\,, & a' \\ b\,, & d\,, & b' \\ a', & c', & a'' \end{vmatrix} + (n-2)\begin{vmatrix} a\,, & b\,, & b' \\ b\,, & c\,, & c' \\ a', & b', & b'' \end{vmatrix}$$

the first of which vanishes. On the right-hand side $D\Lambda=0$ identically, and therefore $D.\Lambda^2 H=\Lambda^2 DH$, or we have

$$(n-2)\begin{vmatrix} a, & c, & a' \\ b, & d, & b' \\ a', & c', & a'' \end{vmatrix}+(n-2)\begin{vmatrix} a, & b, & b' \\ b, & c, & c' \\ a', & b', & b'' \end{vmatrix}=\Lambda^2 DH.$$

16. I repeat the operation D: we have

$$\left.\begin{array}{l}(n-2)(n-1)\begin{vmatrix} b, & c, & a' \\ c, & d, & b' \\ b', & c', & a'' \end{vmatrix}+(n-2)(n-1)\begin{vmatrix} b, & b, & b' \\ c, & c, & c' \\ b', & b', & b'' \end{vmatrix} \\ +(n-2)(n-3)\begin{vmatrix} a, & d, & a \\ b, & c, & b \\ a', & d', & a'' \end{vmatrix}+(n-2)(n-2)\begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix} \\ +(n-2)(n-2)\begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix}+(n-2)(n-3)\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix}\end{array}\right\}=\Lambda^2 D^2 H;$$

or collecting the different terms,

$$(n-2)(n-3)\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix}+2(n-2)^2\begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix}+(n-2)(n-3)\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix}+(n-2)(n-1)\begin{vmatrix} b, & c, & a' \\ c, & d, & b' \\ b', & c', & a'' \end{vmatrix}=\Lambda^2 D^2 H.$$

17. A little consideration will show that in this equation we may write $n-1$ for $n$, and $H_1$ for H. In fact, putting for a moment $\delta=x_{\prime}\partial_x+y_{\prime}\partial_y+z_{\prime}\partial_z$, we have corresponding to the equation

$$\begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix}=\Lambda^2 H,$$

this other equation,

$$\begin{vmatrix} \delta a, & \delta b, & \delta a' \\ \delta b, & \delta c, & \delta b' \\ \delta a', & \delta b', & \delta a'' \end{vmatrix}=\Lambda^2 H_1,$$

where ultimately $(x_{\prime}, y_{\prime}, z_{\prime})$ are to be replaced by $(x, y, z)$. We may operate upon this equation with D, $D^2$, ... as before, the only difference being that in the first instance $\delta a$, $\delta b$, &c. are as regards $(x, y, z)$ of degrees lower by unity than $a$, $b$, &c., that is $n-1$ must be substituted throughout in the place of $n$; and when at the end of the process $(x_{\prime}, y_{\prime}, z_{\prime})$ are replaced by $(x, y, z)$, then $\delta a$, $\delta b$, &c. become equal to $a$, $b$, &c., from which the truth of the asserted proposition is manifest.

18. Hence writing $n=4$, we have

$$2\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix} +8\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix} +2\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix} +6\begin{vmatrix} b, & c, & a' \\ c, & d, & b' \\ b', & c', & a'' \end{vmatrix} = \Lambda^2\mathrm{D}^2\mathrm{H},$$

$$2\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a, & d', & a'' \end{vmatrix} +2\begin{vmatrix} b, & c, & a' \\ c, & d, & b' \\ b', & c', & a'' \end{vmatrix} = \Lambda^2\mathrm{D}^2\mathrm{H}_1;$$

and hence

$$2\left\{\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix} + \begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix} + \begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix}\right\} = \Lambda^2(\mathrm{D}^2\mathrm{H}-3\mathrm{D}^2\mathrm{H}_1),$$

which is the required equation,

$$\mathrm{III}=-\tfrac{1}{2}\Lambda^2(\mathrm{D}^2\mathrm{H}-3\mathrm{D}^2\mathrm{H}_1).$$

19. It is to be added, that the equation for $\Lambda^2\mathrm{DH}$ gives $\mathrm{IV}=\frac{1}{2}\Lambda^2\mathrm{DH}$; the values of II and I are at once obtained from those of III and IV by interchanging $(x, y, z)$ and (X, Y, Z). Hence if we represent by $\mathfrak{H}$, $\mathfrak{D}$, &c. the values which H, D, &c. assume by this interchange, we may write

$$\begin{aligned} \mathrm{I}&=-\tfrac{1}{2}\Lambda^2\mathfrak{D}\mathfrak{H}, \\ \mathrm{II}&=+\tfrac{1}{2}\Lambda^2(\mathfrak{D}^2\mathfrak{H}-3\mathfrak{D}^2\mathfrak{H}_1), \\ \mathrm{III}&=-\tfrac{1}{2}\Lambda^2(\mathrm{D}^2\mathrm{H}-3\mathrm{D}^2\mathrm{H}_1), \\ \mathrm{IV}&=+\tfrac{1}{2}\Lambda^2\mathrm{DH}; \end{aligned}$$

and the identical equation,

$$\mathrm{I}a+\mathrm{II}b+\mathrm{III}d+\mathrm{IV}e=0,$$

gives therefore

$$-\mathfrak{D}\mathfrak{H}\,.\,\mathrm{U}+\tfrac{1}{4}(\mathfrak{D}^2\mathfrak{H}-3\mathfrak{D}^2\mathfrak{H}_1)\mathrm{DU}-\tfrac{1}{4}(\mathrm{D}^2\mathrm{H}-3\mathrm{D}^2\mathrm{H}_1)\mathfrak{D}\Upsilon+\mathrm{DH}\,.\,\Upsilon=0,$$

which is of itself sufficient to put in evidence the property that the curve $\mathrm{D}^2\mathrm{H}-3\mathrm{D}^2\mathrm{H}_1=0$ gives by its intersections with the tangent $\mathrm{DU}=0$, the tangentials of the point $(x, y, z)$. The last-mentioned equation is the equation for a quartic corresponding to Mr. SALMON'S equation

$$-\mathfrak{H}\,.\,\mathrm{U}+\tfrac{1}{3}\mathfrak{D}\mathfrak{H}\,.\,\mathrm{DU}-\tfrac{1}{3}\mathrm{DH}\,.\,\mathfrak{D}\Upsilon+\mathrm{H}\,.\,\Upsilon=0$$

for the cubic $\mathrm{U}=0$.

20. It is worth while to give the investigation of the equation for the cubic; the matrix is

$$\begin{pmatrix} a, & b, & c; & a', & b' \\ b, & c, & d; & b', & c' \\ a', & b', & c'; & a'', & b'' \end{pmatrix}$$

and the identical equation is

$$\mathrm{I}a+\mathrm{II}b+\mathrm{III}c+\mathrm{IV}d=0,$$

where

$$\mathrm{I}=\begin{vmatrix} c, & b, & b' \\ d, & c, & c' \\ c', & b', & b'' \end{vmatrix},$$

$$\mathrm{II}=-\begin{vmatrix} c, & b, & a' \\ d, & c, & b' \\ c', & b', & a'' \end{vmatrix} - \begin{vmatrix} c, & a, & b' \\ d, & b, & c' \\ c', & a', & b'' \end{vmatrix},$$

$$\mathrm{III}=-\begin{vmatrix} a, & b, & b' \\ b, & c, & c' \\ a', & b', & b'' \end{vmatrix} - \begin{vmatrix} a, & c, & a' \\ b, & d, & b' \\ a', & c', & a'' \end{vmatrix},$$

$$\mathrm{IV}=\begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix},$$

or, as we may express them,

$$\begin{aligned}
\mathrm{I}= \quad & a(\ 0\ ) \\
& +b(\ c'^2\ ) && (12) \\
& +c(b''c-b'c'-c'b') && (13) \\
& +d(-b''b+b'b'\ ) && (14) \\
\mathrm{II}= \quad & a(\ -c'^2\ ) && (21) \\
& +b(\ 0\ ) && . \\
& +c(-a''c-b''b+a'c'+b'b'+c'a') && (23) \\
& +d(\ a''b+b''a-a'b'-b'a'\ ) && (24) \\
\mathrm{III}= \quad & a(-b''c+b'c'+c'b'\ ) && (31) \\
& +b(\ a''c+b''b-a'c'-b'b'-c'a') && (32) \\
& +c(\ 0\ ) && . \\
& +d(-aa''\ +a'^2\ ) && (34) \\
\mathrm{IV}= \quad & a(b''b\ -b'b'\ ) && (41) \\
& +b(-a''b-b''a+a'b'+b'a'\ ) && (42) \\
& +c(\ a''a\ -a'a'\ ) && (43) \\
& +d(\ 0\ )
\end{aligned}$$

which verify the identical equation. We have $\mathrm{III}=-\Lambda^2\mathrm{DH}$, $\mathrm{IV}=\Lambda^2\mathrm{H}$, and thence $\mathrm{II}=+\Lambda^2\mathfrak{D}\mathfrak{H}$, $\mathrm{I}=-\Lambda^2\mathfrak{H}$, and thence the equation in question,

$$-\mathfrak{H}.\mathrm{U}+\tfrac{1}{3}\mathfrak{D}\mathfrak{H}.\mathrm{DU}-\tfrac{1}{3}\mathrm{DH}.\mathfrak{D}\Upsilon+\mathrm{H}.\Upsilon=0.$$

21. One other example will be sufficient to render manifest the law of the formation of the multipliers I, II, III, IV.

In the case of a sextic curve we have the matrix

$$\begin{pmatrix} a, & b, & c, & d, & e, & f; & a', & b', & c', & d', & e' \end{pmatrix},$$
$$\begin{vmatrix} b, & c, & d, & e, & f, & g; & b', & c', & d', & e', & f' \\ a', & b', & c', & d', & e', & f'; & a'', & b'', & c'', & d'', & e'' \end{vmatrix}$$

the identical equation is

$$\text{I}a+\text{II}b+\text{III}f+\text{IV}g=0;$$

and the expressions for the multipliers I, II, III, IV are—

$$\text{I}=\begin{vmatrix} f, & e, & b' \\ g, & f, & c' \\ f', & e', & b'' \end{vmatrix}+\begin{vmatrix} f, & d, & c' \\ g, & e, & d' \\ f', & d', & c'' \end{vmatrix}+\begin{vmatrix} f, & c, & d \\ g, & d, & e \\ f', & c', & d'' \end{vmatrix}+\begin{vmatrix} f, & b, & e' \\ g, & c, & f' \\ f', & b', & e'' \end{vmatrix},$$

$$\text{II}=-\begin{vmatrix} f, & e, & a' \\ g, & f, & b' \\ f', & e', & a'' \end{vmatrix}-\begin{vmatrix} f, & d, & b' \\ g, & e, & c' \\ f', & d', & b'' \end{vmatrix}-\begin{vmatrix} f, & c, & c' \\ g, & d, & d' \\ f', & c', & d'' \end{vmatrix}-\begin{vmatrix} f, & b, & d' \\ g, & c, & e' \\ f', & b', & d'' \end{vmatrix}-\begin{vmatrix} f, & a, & e' \\ g, & b, & f' \\ f', & a', & e'' \end{vmatrix},$$

$$\text{III}=-\begin{vmatrix} a, & b, & e' \\ b, & c, & f' \\ a', & b', & e'' \end{vmatrix}-\begin{vmatrix} a, & c, & d' \\ b, & d, & e' \\ a', & c', & e'' \end{vmatrix}-\begin{vmatrix} a, & d, & c' \\ b, & e, & d' \\ a', & d', & e'' \end{vmatrix}-\begin{vmatrix} a, & e', & b' \\ b, & f', & c' \\ a', & e'', & b'' \end{vmatrix}-\begin{vmatrix} a, & f, & a' \\ b, & g, & b' \\ a', & f', & a'' \end{vmatrix},$$

$$\text{IV}=\begin{vmatrix} a, & b, & d' \\ b, & c, & e' \\ a', & b', & d'' \end{vmatrix}+\begin{vmatrix} a, & c, & c' \\ b, & d, & d' \\ a', & d', & c'' \end{vmatrix}-\begin{vmatrix} a, & d, & b' \\ b, & e, & c' \\ a', & d', & b'' \end{vmatrix}-\begin{vmatrix} a, & e, & a' \\ b, & f, & b' \\ a', & e', & a'' \end{vmatrix}.$$

22. We have in fact

$$\begin{array}{lllll}
\text{I}= & a\ 0 & & & \\
 & +b(-ge'' & +f'f' & ) & (12) \\
 & +f(\ \ b''f+c''e+d''d+e''c & -b'f'-c'e'-d'd'-e'c'-f'b' & ) & (13) \\
 & +g(-b''e-c''d-d''c & +b'e'+c'd'+d'c'+e'b' & ) & (14) \\
\text{II}= & a(\ \ e''g & -f'f' & ), & (21) \\
 & +b\ 0 & & & \\
 & \multicolumn{3}{l}{+f(-a''f-b''e-c''d-d''e-e''b+a'f'+b'e'+c'd'+d'c'+e'b'+f'a')} & (23) \\
 & +g(\ \ a''e+b''d+c''c+d''b & -a'e'-b'd'-c'c'-d'b'-e'a' & ) & (24) \\
\text{III}= & a(-b''f-c''e-d''d-e''c & +b'f'+c'e'+d'd'+e'c'+f'b' & ) & (31) \\
 & \multicolumn{3}{l}{+b(\ \ a''f+b''e+c''d+d''c+e''b-a'f'-b'e'-c'd'-d'c'-e'b'-f'a')} & (32) \\
 & +f\ 0 & & & \\
 & +g(-a''a & +a'^2 & ), & (34)
\end{array}$$

$$\begin{array}{lll} \text{IV}=a( \;\; b''e+c''d+d''c & -b'e'-c'd'-d'c'-e'b' & ) \quad (41) \\ +b(-a''e-b''d-c''c-d''b & +a'e'+b'd'+c'c'+d'b'+e'a' & ) \quad (42) \\ +f( \;\; a''a & -a'^2 & ) \quad (43) \\ +g \;\; 0 & & , \end{array}$$

which are of the form

$$\begin{aligned} \text{I}&=a\;0\;+b(12)+f(13)+g(14), \\ \text{II}&=a(21)+b\;0\;+f(23)+g(24), \\ \text{III}&=a(31)+b(32)+f\;0\;+g(34), \\ \text{IV}&=a(41)+b(42)+f(43)+g\;0, \end{aligned}$$

where (12)=−(21) &c., and the equation

$$\text{I}a+\text{II}b+\text{III}f+\text{IV}g=0$$

is consequently satisfied.

23. The expression

$$\text{III}=-\begin{vmatrix} a, & b, & e' \\ b, & c, & f' \\ a', & b', & e'' \end{vmatrix}-\begin{vmatrix} a, & c, & d' \\ b, & d, & e' \\ a', & c', & d'' \end{vmatrix}-\begin{vmatrix} a, & d, & c' \\ b, & e, & d' \\ a', & d', & c'' \end{vmatrix}-\begin{vmatrix} a, & f, & a' \\ b, & g, & b' \\ a', & f', & a'' \end{vmatrix}$$

leads to

$$\text{III}=-\Lambda^2(\text{D}^4\text{H}-5\text{D}^4\text{H}_1+10\text{D}^4\text{H}_2),$$

and consequently the equation of the curve which by its intersections with the tangent determines the tangentials of a point of a sextic, is

$$\text{D}^4\text{H}-5\text{D}^4\text{H}_1+10\text{D}^4\text{H}_2=0.$$

24. In the general case of a curve of the order $n$ the matrix is

$$\left(\begin{array}{lllll|llll} a_0, & a_1, & a_2 & \ldots & a_{n-1}; & a'_0, & a'_1 & \ldots & a'_{n-2} \\ a_1, & a_2, & a_3 & \ldots & a_n \;\; ; & a'_1, & a'_2 & \ldots & a'_{n-1} \\ a'_0, & a'_1, & a'_2 & \ldots & a'_{n-1}; & a''_0, & a''_1 & \ldots & a''_{n-2} \end{array}\right)$$

where, in analogy with what precedes,

$$\begin{aligned} a_0 \;&=(\text{X}, \text{Y}, \text{Z})^n \qquad\qquad , \\ a_1 \;&=(\text{X}, \text{Y}, \text{Z})^{n-1}(x, y, z), \\ &\;\;\cdot \\ &\;\;\cdot \\ a_{n-1}&=(\text{X}, \text{Y}, \text{Z}) \quad (x, y, z)^{n-1}, \\ a_n \;&= \qquad\qquad\quad\; (x, y, z)^n \;\;, \end{aligned}$$

and similarly for the accented letters, so that

$a_0 \;=0$ is the equation of the curve;

$a_1 \;=0$ is the equation of the first or $(n-1)$thic polar;

$a_{n-1}=0$ is the equation of the last or line-polar, or what is the same thing, since $(x, y, z)$ is a point on the curve, the tangent at this point;

$a_n \;=0$, the condition which expresses that $(x, y, z)$ is a point of the curve;

and we have to form the identical equation

$$\mathrm{I}a_0+\mathrm{II}a_1+\mathrm{III}a_{n-1}+\mathrm{IV}a_n=0.$$

25. If, for shortness, the columns of the last-mentioned matrix are represented by

$$1,\ 2,\ 3\ \ldots\ n,\ (1),\ (2)\ ..\ (n-1),$$

and the determinants formed with these columns respectively by a corresponding notation $\{1, 2, (1)\}$, $\{1, 2, (2)\}$, &c., then the expressions for the multipliers I, II, III, IV are as follows, viz.

$$\begin{aligned}
\mathrm{I}&= \{n, n-1, (2)\}+\{n, n-2, (3)\}\ldots+\{n, 2, (n-1)\} ,\\
\mathrm{II}&=-\{n, n-1, (1)\}-\{n, n-2, (2)\}\ldots-\{n, 2, (n-2)\}-\{n, 1, (n-1)\},\\
\mathrm{III}&=-\{1, 2, (n-1)\}-\{1, 3, (n-2)\}\ldots-\{1, n-1, (2)\}-\{1, n, (1)\},\\
\mathrm{IV}&= \{1, 2, (n-2)\}+\{1, 3, (n-1)\}\ldots+\{1, n-1, (1)\};
\end{aligned}$$

the truth of the identical equation being shown, as in the foregoing special cases, by the transformation of the multipliers into the form

$$\begin{aligned}
\mathrm{I}&=a_0\ 0\ +a_1(12)+a_{n-1}(13)+a_n(14),\\
\mathrm{II}&=a_0(21)+a_1\ 0\ +a_{n-1}(23)+a_n(24),\\
\mathrm{III}&=a_0(31)+a_1(32)+a_{n-1}\ 0\ +a_n(34),\\
\mathrm{IV}&=a_0(41)+a_1(42)+a_{n-1}(43)+a_n\ 0\ ,
\end{aligned}$$

where $(12)=-(21)$ &c.: the required expressions may be written down without difficulty.

26. Proceeding then to reduce the equation

$$\mathrm{III}=-\{1,2,(n-1)\}-\{1,3,(n-2)\}\ldots-\{1,n-1,(2)\}-\{1,n,(1)\},$$

we have the equation

$$\{1,2,(1)\}=\Lambda^2\mathrm{H},$$

which is to be successively operated on with D. The degrees (less unity) of the columns

$$1,\quad 2,\ ..\ n-1,\ n,\ (1),\quad (2),\ ..\ (n-1),$$

are

$$n-1,\ n-2,\ ..\ 1,\qquad 0,\ n-2,\ n-3,\ ..\ 0;$$

and the rule is to operate on each column of the determinant, multiplying by the degree less unity, and increasing the symbolical number by unity. Thus

$$\begin{aligned}
\mathrm{D}\{1, 2, (1)\}&=(n-1)\{2, 2, (1)\}+(n-2)\{1, 3, (1)\}+(n-2)\{1, 2, (2)\}\\
&=(n-2)\{1, 3, (1)\}+(n-2)\{1, 2, (2)\},
\end{aligned}$$

since $\{2, 2, (1)\}$ vanishes identically. The following Table shows the mode of effecting the operations:—

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| H | 1 | | | = 1 | $[n-2]^0$ | $[n-2]^0$ | | 12(1) |
| DH | 1 | | | = 1 | $[n-2]^0$ | $[n-2]^1$ | | 12(2) |
| | | 1 | | 1 | $[n-2]^1$ | $[n-2]^0$ | | 13(1) |
| D²H | 1 | | | = 1 | $[n-2]^0$ | $[n-2]^2$ | | 12(3) |
| | 1 | 1 | | 2 | $[n-2]^1$ | $[n-2]^1$ | | 13(2) |
| | | 1 | | 1 | $[n-2]^2$ | $[n-2]^0$ | | 14(1) |
| | | | 1 | 1 | $[n-2]^1$ | $[n-2]^0$ | $[n-1]^1$ | 23(1) |
| D³H | 1 | | | 1 | $[n-2]^0$ | $[n-2]^3$ | | 12(4) |
| | 2 | 1 | | 3 | $[n-2]^1$ | $[n-2]^2$ | | 13(3) |
| | 1 | 2 | | 3 | $[n-2]^2$ | $[n-2]^1$ | | 14(2) |
| | | 1 | | 1 | $[n-2]^3$ | $[n-2]^0$ | | 15(1) |
| | 1 | | 2 | 3 | $[n-2]^1$ | $[n-2]^1$ | } $[n-1]^1$ | 23(2) |
| | | 1 | 1 | 2 | $[n-2]^2$ | $[n-2]^0$ | | 24(1) |
| D⁴H | 1 | | | = 1 | $[n-2]^0$ | $[n-2]^4$ | | 12(5) |
| | 3 | 1 | | 4 | $[n-2]^1$ | $[n-2]^3$ | | 13(4) |
| | 3 | 3 | | 6 | $[n-2]^2$ | $[n-2]^2$ | | 14(3) |
| | 1 | 3 | | 4 | $[n-2]^3$ | $[n-2]^1$ | | 15(2) |
| | | 1 | | 1 | $[n-2]^4$ | $[n-2]^0$ | | 16(1) |
| | 3 | | 3 | 6 | $[n-2]^1$ | $[n-2]^2$ | | 23(3) |
| | 2 | 3 | 3 | 8 | $[n-2]^2$ | $[n-2]^1$ | } $[n-1]^1$ | 24(2) |
| | | 2 | 1 | 3 | $[n-2]^3$ | $[n-2]^0$ | | 25(1) |
| | | | 2 | 2 | $[n-2]^2$ | $[n-2]^0$ | $[n-1]^2$ | 34(1) |
| D⁵H | 1 | | | = 1 | $[n-2]^0$ | $[n-2]^5$ | | 12(6) |
| | 4 | 1 | | 5 | $[n-2]^1$ | $[n-2]^4$ | | 13(5) |
| | 6 | 4 | | 10 | $[n-2]^2$ | $[n-2]^3$ | | 14(4) |
| | 4 | 6 | | 10 | $[n-2]^3$ | $[n-2]^2$ | | 15(3) |
| | 1 | 4 | | 5 | $[n-2]^4$ | $[n-2]^1$ | | 16(2) |
| | | 1 | | 1 | $[n-2]^5$ | $[n-2]^0$ | | 17(1) |
| | 6 | | 4 | 10 | $[n-2]^1$ | $[n-2]^4$ | | 23(4) |
| | 8 | 6 | 6 | 20 | $[n-2]^2$ | $[n-2]^3$ | } $[n-1]^1$ | 24(3) |
| | 3 | 8 | 4 | 15 | $[n-2]^3$ | $[n-2]^2$ | | 25(2) |
| | | 3 | 1 | 4 | $[n-2]^4$ | $[n-2]^1$ | | 26(1) |
| | 2 | | 8 | 10 | $[n-2]^2$ | $[n-2]^1$ | } $[n-1]^2$ | 34(2) |
| | | 2 | 3 | 5 | $[n-2]^3$ | $[n-2]^0$ | | 35(1) |
| D⁶H | 1 | | | = 1 | $[n-2]^0$ | $[n-2]^6$ | | 12(7) |
| | 5 | 1 | | 6 | $[n-2]^1$ | $[n-2]^5$ | | 13(6) |
| | 10 | 5 | | 15 | $[n-2]^2$ | $[n-2]^4$ | | 14(5) |
| | 10 | 10 | | 20 | $[n-2]^3$ | $[n-2]^3$ | | 15(4) |
| | 5 | 10 | | 15 | $[n-2]^4$ | $[n-2]^2$ | | 16(3) |
| | 1 | 5 | | 6 | $[n-2]^5$ | $[n-2]^1$ | | 17(2) |
| | | 1 | | 1 | $[n-2]^6$ | $[n-2]^0$ | | 18(1) |
| | 10 | | 5 | 15 | $[n-2]^2$ | $[n-2]^4$ | | 23(5) |
| | 20 | 10 | 10 | 40 | $[n-2]^2$ | $[n-2]^3$ | | 24(4) |
| | 15 | 20 | 10 | 45 | $[n-2]^3$ | $[n-2]^2$ | } $[n-1]^1$ | 25(3) |
| | 4 | 15 | 5 | 24 | $[n-2]^4$ | $[n-2]^1$ | | 26(2) |
| | | 4 | 1 | 5 | $[n-2]^5$ | $[n-2]^0$ | | 27(1) |
| | 10 | | 20 | 30 | $[n-2]^2$ | $[n-2]^2$ | | 34(3) |
| | 5 | 10 | 15 | 30 | $[n-2]^3$ | $[n-2]^1$ | } $[n-1]^2$ | 35(2) |
| | | 5 | 4 | 9 | $[n-2]^4$ | $[n-2]^0$ | | 36(1) |
| | | | 5 | 5 | $[n-2]^3$ | $[n-2]^0$ | $[n-1]^3$ | 45(1) |
| D⁷H | &c. | | | | | | | |

where the first three columns show the numbers which give, by the addition of the numbers in the same horizontal line, the numerical coefficients of the factorials which multiply the different terms of H, DH, &c., and where in the last column 12(1), &c. are written for shortness in the place of $\{1, 2, (1)\}$, &c.

27. It is clear that we have in general

$$
\begin{array}{lllll}
A^2D^rH = & & 1 & [n-2]^0[n-2]^r & \{1,\ 2,\ (r+1)\} \\
 & + & \frac{r}{1} & [n-2]^1[n-2]^{r-1} & \{1,\ 3,\ (r)\} \\
 & + & \frac{r.r-1}{1.2} & [n-2]^2[n-2]^{r-2} & \{1,\ 4,\ (r-1)\} \\
 & & \vdots & & \\
 & + & 1 & [n-2]^r[n-2]^0 & \{1,\ r+2,\ (1)\} \\
\hline
+[n-1]^1 & \left\{\begin{array}{l} +R'_0 \\ +R'_1 \\ \vdots \\ +R'_{r-2} \end{array}\right. & & \begin{array}{l} [n-2]^1\ [n-2]^{r-2} \\ [n-2]^2\ [n-2]^{r-3} \\ \\ [n-2]^{r-1}[n-2]^0 \end{array} & \begin{array}{l} \{2,\ 3,\ (r-1)\} \\ \{2,\ 4,\ (r-2)\} \\ \\ \{2,\ r+1,\ (1)\} \end{array} \\
\hline
+[n-1]^2 & \left\{\begin{array}{l} +R''_0 \\ \vdots \\ +R''_{r-4} \end{array}\right. & & \begin{array}{l} [n-2]^2\ [n-2]^{r-4} \\ \\ [n-2]^{r-2}[n-2]^0 \end{array} & \begin{array}{l} \{3,\ 4,\ (r-3)\} \\ \\ \{3,\ r,\ (1)\} \end{array} \\
\hline
 & \vdots & & & \\
+[n-1]^{\frac{1}{2}r} & \left\{\begin{array}{l} +R_0^{\frac{1}{2}r} \\ \quad r \text{ even; or } r \text{ odd} \end{array}\right. & & [n-2]^{\frac{1}{2}r}\ [n-2]^0 & \{\tfrac{1}{2}r+1,\ \tfrac{1}{2}r+2,\ (1)\} \\
+[n-1]^{\frac{1}{2}(r-1)} & \left\{\begin{array}{l} +R_0^{\frac{1}{2}(r-1)} \\ +R_0^{\frac{1}{2}(r+1)} \end{array}\right. & & \begin{array}{l} [n-2]^{\frac{1}{2}(r-1)}[n-2]^1 \\ [n-2]^{\frac{1}{2}(r+1)}[n-2]^0 \end{array} & \begin{array}{l} \{\tfrac{1}{2}(r+1),\ \tfrac{1}{2}(r+3),\ (2)\} \\ \{\tfrac{1}{2}(r+1),\ \tfrac{1}{2}(r+5),\ (1)\}\ ; \end{array}
\end{array}
$$

and the general term is

$$[n-1]^\delta\ R_s^\delta[n-2]^{\delta+s}[n-2]^{r-2\delta-s}\{\delta+1,\ \delta+2+s,\ (r-2\delta+1-s)\},$$

where $s$ extends from $s=0$ to $s=r-2\delta$, and $\delta$ from $\delta=0$ to $\delta=\frac{1}{2}r$ or $\frac{1}{2}(r-1)$, according as $r$ is even or odd. The expression for the coefficients $R_s^0$ is

$$R_s^0=\frac{[r]^s}{[s]^s},$$

and that of the other coefficients $R_s^\delta$ ($\delta=$ or $<1$) is not required for the present purpose.

28. According to a remark already made, the expressions for $D^rH_1$, $D^rH_2$ &c. are at once obtained from that for $D^rH$ by merely writing $n-1$, $n-2$, &c. in the place of $n$: it is however to be noticed, that the quantity within the [ ] must not be negative, and that on its becoming so, the factorial is to be omitted.

29. I write now

$$+s=\alpha,\quad r-2\delta-s=\beta,$$

and I consider the expression

$$D^r H - \frac{n-1}{1} D^r H_1 + \&c.;$$

the general term of which is

$$R_s^\delta\{\delta+1,\ \delta+2+s,\ (r-2\delta+1-s)\} \times \left\{\begin{array}{l} \qquad [n-1]^\delta[n-2]^\alpha[n-2]^\beta \\ -\frac{n-1}{1}[n-2]^\delta[n-3]^\alpha[n-3]^\beta \\ + \qquad \&c. \end{array}\right\},$$

or as this may be written, putting $q=n-\delta-1$,

$$R_s^\delta\{\delta+1,\ \delta+2+s,\ (r-2\delta+1-s)\} \times [n-1]^\delta \left\{\begin{array}{l} \qquad [n-2]^\alpha[n-2]^\beta \\ -\frac{q}{1}[n-3]^\alpha[n-3]^\beta \\ \quad + \ \&c. \end{array}\right\}.$$

30. I assume $r \not> n-2$, we have then $\alpha+\beta=r-\delta \not> n-\delta-2$, and therefore $\alpha+\beta<q$. The general term of the series in {} is

$$(-)^\vartheta \frac{[q]^\vartheta}{[\vartheta]^\vartheta}[n-2-\vartheta]^\alpha[n-2-\vartheta]^\beta,$$

where the terms for which $n-2-\vartheta$ is negative are to be excluded, or what is the same thing, the series is not to be continued beyond $\vartheta=n-2$. But observing that $[q]^\vartheta$ vanishes for $\vartheta>q$, that is, $\vartheta>n-\delta-1$, it is in fact the same thing whether the series is continued indefinitely or only to the term for which $\vartheta=n-\delta-1$, and we may consistently with the condition $\vartheta \not> n-2$, continue the series as far as $\vartheta=n-\delta-1$, except in the case $\delta=0$, when by doing so we include the term corresponding to $\vartheta=n-1$, which in virtue of the condition ought to be excluded. The expression for the term in question is $(-)^{n-1}[-1]^\alpha[-1]^\beta$; hence if the sum of the series continued to the proper point is S, the sum continued indefinitely (in the particular case $\delta=0$) is $S+(-)^{n-1}[-1]^\alpha[-1]^\beta$, but in every other case the sum continued indefinitely is simply S. And by a well-known theorem in finite differences, the sum continued indefinitely is in fact zero. That is, except in the case $\delta=0$, we have

$$S=0,$$

but in the excepted case

$$S+(-)^{n-1}[-1]^\alpha[-1]^\beta=0;$$

or observing that $\alpha+\beta(=r-\delta)$ is in this case $=r$, and transforming the factorials, we have

$$S=(-)^{n-r}[\alpha]^\alpha[\beta]^\beta,$$

or substituting for $\alpha$ and $\beta$ their values,

$$S=(-)^{n-r}[s]^s[r-s]^{r-s}.$$

31. Hence the general term of

$$D^rH-\frac{n-1}{1}D^rH_1+\&c.$$

vanishes except for $\delta=0$, but when $\delta=0$, its value is

$$R^0_s\{1, 2+s, (r+1-s)\}\times(-)^{n-r}[s]^s[r-s]^{r-s};$$

or observing that $R^0_s$ is equal to $[r]^r\div[s]^s[r-s]^{r-s}$, the value is simply

$$(-)^{n-r}[r]^r\{1, 2+s, (r+1-s)\},$$

that is, we have

$$D^rH-\frac{n-1}{1}D^rH_1+\&c.$$
$$=(-)^{n-r}[r]^r\,S_s\{1, 2+s, (r+1-s)\},$$

the summation in respect to $s$ extending from $s=0$ to $s=r$. In particular, giving to $r$ the values $n-2$ and $n-1$, and attending to the expressions for III and IV, we find

$$\Lambda^2\left(D^{n-2}H-\frac{n-1}{1}D^{n-2}H_1+\&c.\,..\right)=-[n-2]^{n-2}\,\text{III},$$

$$\Lambda^2\left(D^{n-3}H-\frac{n-1}{1}D^{n-3}H_1+\&c.\,..\right)=-[n-3]^{n-3}\,\text{IV}.$$

32. The equation III$=0$ belongs to the curve which by its intersections with the tangent, gives the tangentials of a point of the curve $U=0$. Hence the equation of the curve in question is

$$D^{n-2}H-\frac{n-1}{1}D^{n-2}H_1+\&c.=0,$$

which is Mr. SALMON's theorem, leading to the solution of the problem of double tangents.

33. The expressions for I and II are obtained from those of IV and III by interchanging (X, Y, Z) and $(x, y, z)$, and reversing the sign. Hence if, as before, $\mathfrak{H}$, $\mathfrak{D}$, &c. denote the values which H, D, &c. assume by this interchange, we have

$$\Lambda^2\left(\mathfrak{D}^{n-2}\mathfrak{H}-\frac{n-1}{1}\mathfrak{D}^{n-2}H_1+\&c.\,...\right)=[n-2]^{n-2}\,\text{II},$$

$$\Lambda^2\left(\mathfrak{D}^{n-3}\mathfrak{H}-\frac{n-1}{1}\mathfrak{D}^{n-3}H_1+\&c.\,...\right)=[n-3]^{n-3}\,\text{I},$$

and the identical equation

$$\text{I}a_0+\text{II}a_1+\text{III}a_{n-1}+\text{IV}a_n=0$$

becomes therefore

$$\left.\begin{aligned}&\left(\mathfrak{D}^{n-3}\mathfrak{H}-\frac{n-1}{1}\mathfrak{D}^{n-3}\mathfrak{H}_1+\&c.\right)U\\ &+\frac{1}{n(n-2)}\left(\mathfrak{D}^{n-2}\mathfrak{H}-\frac{n-1}{1}\mathfrak{D}^{n-2}\mathfrak{H}_1+\&c.\right)DU\\ &-\frac{1}{n(n-2)}\left(D^{n-2}H-\frac{n-1}{1}D^{n-2}H_1+\&c.\right)\mathfrak{D}\Upsilon\\ &\left(D^{n-3}H-\frac{n-1}{1}D^{n-3}H_1+\&c.\right)\Upsilon\end{aligned}\right\}=0,$$

which is the general identical equation referred to in the introduction to the present memoir.

34. It is to be noticed that for $n=3$, the equation is

$$(\mathfrak{H}-2\mathfrak{H}_1)U+\tfrac{1}{3}\mathfrak{D}\mathfrak{H}.DU-\tfrac{1}{3}DH.\mathfrak{D}\Upsilon-(H-2H_1)\Upsilon=0.$$

But we have $H_1=H$, and in like manner $\mathfrak{H}_1=\mathfrak{H}$, and the equation thus becomes

$$-\mathfrak{H}U+\tfrac{1}{3}\mathfrak{D}\mathfrak{H}.D\Upsilon-\tfrac{1}{3}DH.\mathfrak{D}\Upsilon+H\Upsilon=0.$$

And so also for $n=4$, the equation is

$$(\mathfrak{D}\mathfrak{H}-3\mathfrak{D}\mathfrak{H}_1)U+\tfrac{1}{8}(\mathfrak{D}^2\mathfrak{H}-3\mathfrak{D}^2\mathfrak{H}_1)DU-\tfrac{1}{8}(D^2H-3D^2H_1)\mathfrak{D}\Upsilon-(DH-3DH_1)\Upsilon=0.$$

But we have in general $DH_1=\frac{n-3}{n-2}DH$, and therefore in the present case $DH_1=\frac{1}{2}DH$, and consequently $\mathfrak{D}\mathfrak{H}_1=\frac{1}{2}\mathfrak{D}\mathfrak{H}$, and the equation thus becomes

$$-\mathfrak{D}\mathfrak{H}.U+\tfrac{1}{4}(\mathfrak{D}^2\mathfrak{H}-3\mathfrak{D}^2\mathfrak{H}')DU-\tfrac{1}{4}(D^2H-3D^2H_1)\mathfrak{D}\Upsilon+DH.\Upsilon=0,$$

which agree with the results previously obtained for the two particular cases.